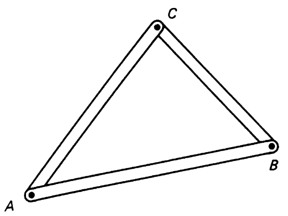
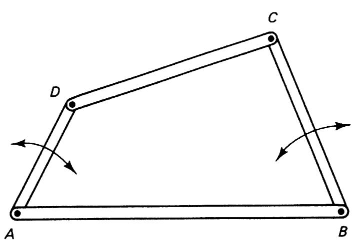
Appendix 10: Examples of applications of geometry

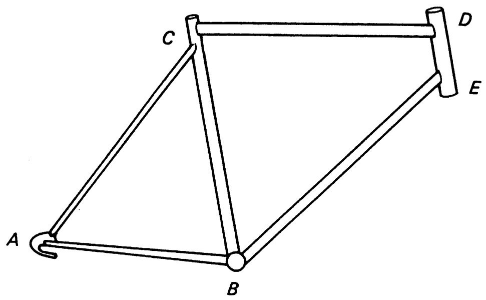
There are many examples to be found in familiar rigid structures and in mechanisms. Many of these have been collected in a form suitable for classroom use by Brian Bolt. The following extract on rigid structures is taken from Brian Bolt's submission to the working group.

1 Rigid structures



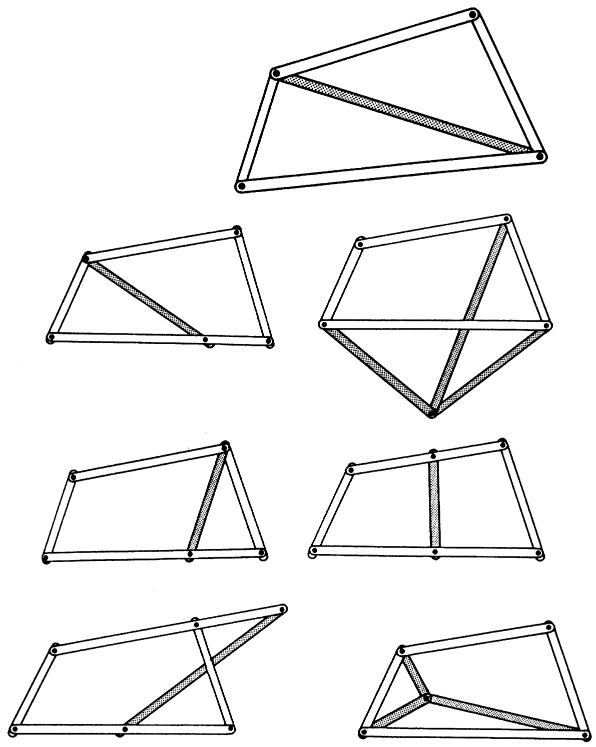
Make a triangle linkage from plastic geostrips or card strips using paper fasteners to join their ends. If you place the linkages on a table and fix AB then the point C or the triangle linkage is also fixed, but points C and D of the quadrilateral linkage are free to move. This illustrates the innate rigidity of the triangle linkage, but lack of rigidity of the quadrilateral linkage. The traditional cycle frame makes use of the triangles strength at the rear, but the main part of the frame BCDE, which supports the front forks, is dependent on the strength of the welds in the joints.



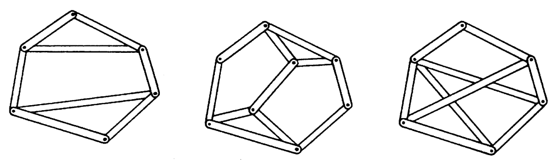


If you are observant you will not have to look far to see places where the rigidity of the triangle is used. Look at a folding chair, the fastening which hold a window open, the roof timbers in a house, an umbrella, a rotating clothes airier, the legs of an ironing board, the design of a traditional 5-bar gate, just to name a few.

To increase understanding of which two-dimensional frameworks are intrinsically rigid an investigation can be made, for example, of what struts can be added to a quadrilateral in order for it to maintain one shape. It is soon clear that one of the diagonals will suffice but how about all the other possibilities suggested by the following drawings. A traditional geometrical education gives little help in this and it is essential to make models to come to terms with the problem.



Having decided which of the above frameworks are rigid the investigation can be followed up by considering pentagonal or hexagonal frameworks. How about the following? Are any of them rigid? What is the minimum number of struts required to ensure that an n-gon framework is rigid?

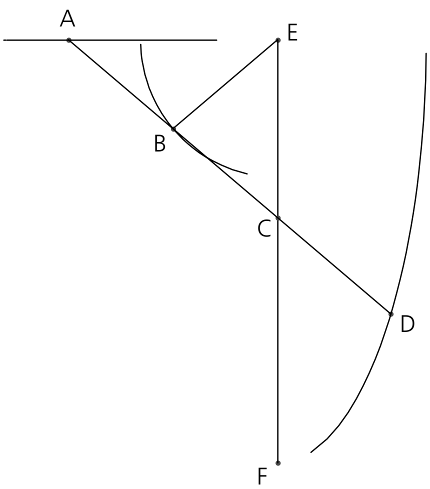


This concept of intrinsically rigid framework should be extended into three dimensions. There are so many structures in our modern society where they can be observed. Tower cranes can be seen like exotic creatures looking down on the buildings being built beneath them. The structures of their towers and their jibs are prime examples of rigid frameworks as are the electrical pylons which stride like giants across the countryside. Many bridge structures such as the famous Forth Bridge in Scotland or the Sydney Harbour Bridge in Australia are well known examples but so are the many railway bridges from the pioneering days when they were made of timber to the metal ones of today. The Eiffel Tower in Paris is such a structure, but on a small scale look at the scaffolding on a building site or visit a fun fair to see exciting examples.

2 Garage door

An example of modelling the mechanism for an 'up and over' garage door is given in [Oldknow and Taylor, 2000]. This provides a good example for the study of locus.

Figure 1

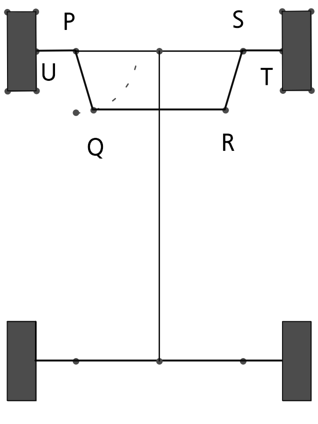


Here the door is represented by *AD*. The bar *EB* is free to rotate about the top of the door frame *E*, and a fixed pin at *B*. The pin at *C* is free to slide inside a groove along *EF*. A model can easily be made using drawing tools, cardboard strips or geometric software. Using dynamic geometry the loci of points such as *A*, *B* and *D* can be drawn. Given that $BA = BE = BC$ geometric reasoning can be used to establish the loci of *A* and *B*. At A-level, coordinate geometry can be used to find the coordinates (x,y) of *D* in terms of the angle $\theta = BEF$ and the lengths $a = AB$ and $b = CD$, and hence to deduce the equation of the locus of *D* as an arc of an ellipse.

3 A car steering mechanism

The following is adapted from *Teaching Mathematics with ICT* by Adrian Oldknow and Ron Taylor.
The system, known as Ackermann steering, is based on a trapezium. When the front wheels are pointing straight ahead, the quadrilateral *PQRS* forms a trapezium. The 'stub axle' *UP* makes a fixed angle with the 'track-rod end' *PQ*, and they both pivot about the fixed point *P* (the 'king-pin'). Similarly for *R,S* and *T*. Points *Q* and *R* are joined by a rod, called the tie bar, which can pivot loosely at *Q* and *R*. If *Q* is moved on a circular arc centre *P*, so *R* describes a circular arc centre *S*. For a given wheelbase *UT* and length between axles, the shape of the trapezium is defined by the two parameters $p = PQ$, the length of the track rod ends, and $q = QR$, the length of the tie bar. As *Q* slides on the arc centre *P*, the stub axles *PU* and *ST* turn through different angles. (They would be the same if *PQRS* was a parallelogram.)

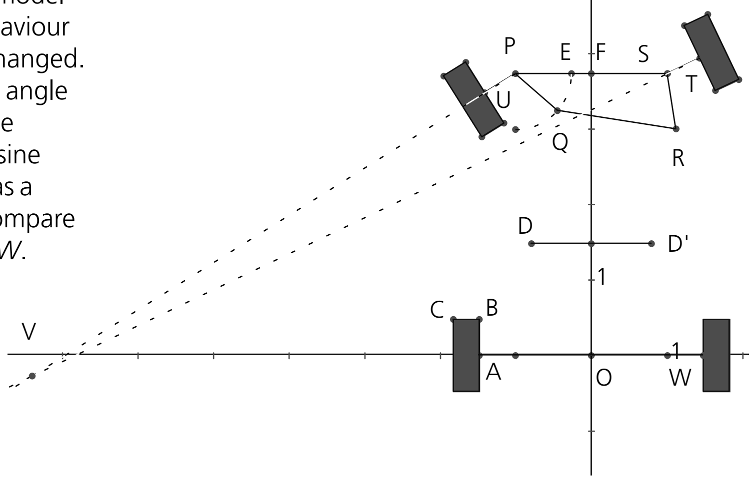
Figure 2



Now it is highly desirable that when taking a bend, the four circles to which the tyres are tangents should all have the same centre - otherwise the front tyres will soon lose their tread. The design problem is to choose *p* and *q* so that the point *V* of intersection of the stub axles produced lies as close to the line *AW* as possible for all positions of *Q*. Of course there also physical constraints on the maximum sizes of *p* and *q*.

The diagram above suggests making a dynamic model e.g. in *Cabri* from which you could study the behaviour of the locus of *V* as the parameters *p* and *q* are changed. You could also make an analytic model using the angle *QPS* = θ as independent variable, and splitting the quadrilateral *PQRS* into two triangles. Using the sine and cosine rules you can find the angle *QRS* = ϕ as a function of θ (perhaps in a spreadsheet?), and compare it with the desired value ϕ' found when *V* is on *AW*.

Figure 3



4 A London day

Now that digital cameras, or scanners for computers, are more or less commonplace a trip out can be used to capture a variety of geometric images which might act as stimuli for work in geometry in schools and colleges. Here are a set of photographs taken of the 'London Eye'. Can you suggest what sort of route the camera operator took while shooting these pictures? Why is it that a circular object appears elliptical when viewed from an angle. Can you find a way to use the ratio between the shortest diameter and the longest diameter of a picture of a circle to work out the angle the picture must have been taken from, measured from the axis of the circle?

5 CAD and Bézier curves

Design tasks used to be carried out using drawing boards. In order to draw smooth curves there was a device called a 'draughtman's spline' - where weights could be placed on the board, with grooves on their top, through which a flexible piece of steel, or laminated wood (called a 'spline') could be passed. As the weights were moved so the flexible curve could be controlled to take a desired shape. With the move to computer based design there have been several systems developed to produce a 'virtual' equivalent to the physical 'flexible curve'. One such fundamental form for representing flexible curves in Computer Aided Design is based on Bézier curves. Here n+1 points are defined on the screen which are used to define a unique n-th degree polynomial. If any point is moved, then the whole curve is changed. The points are called 'control points', and the effect is called 'global control'. Unlike some other systems for producing flexible curves, Bézier curves do not, in general, pass through the control points (other than the first and last). The curves are usually defined algebraically using binomial coefficients. However they can also be constructed by a set of dilations (also known as dilatations).

First we define a parameter between 0 and 1 by taking a point *T* which can slide on a segment *PQ*, and taking the ratio of *PT* to *PQ* as the parameter t (with $0 \leq t \leq 1$). Points *A, B, C, D* etc. are used to define the curve. The diagrams below show the special cases of a quadratic curve defined by three control points *A,B,C* and a cubic curve defined by adding a fourth control point *D*. The point *B′* on *AB* is the image of *B* when dilated with centre *A* and scale factor t. The points *C′* and *D′* are similarly defined. The point *C″* on *B′C′* is the image of *C′* when dilated with centre *B′* and scale factor. The locus of *C″* as *T* slides on *PQ* is the desired quadratic. Points *D″* and *D‴* are similarly defined, and the locus of *D‴* is the desired cubic.

The quadratic curve starts at *A*, tangent to *AB* and finishes at *C* tangent to *BC*. If a is the position vector of *A*, etc. then $b' = (1-t)a + tb$, and similarly for c′ and d′. Hence $c'' = (1-t)b' + tc'$ from where you can show that the locus of *C″* is quadratic in the parameter t, and deduce the claims about tangency. What can you say about the locus of *D″* shown above? Now define *D‴* as the image of *D″* when dilated with centre *C″* and scale factor t. The position vector of *D‴* is the weighted average of the position vectors of *C″* and *D″* and so its locus is a cubic. Can you find its form in terms of t ? *A, B, C* and *D* need not be coplanar, and hence the locus of *D‴* can represent a 'twisted' curve in space, such as a section of a car's exhaust system. Cubic curves are frequently used as the basis for design systems since they are the simplest polynomials which can exhibit inflections.