Appendix 13: Integrated approaches to geometry teaching

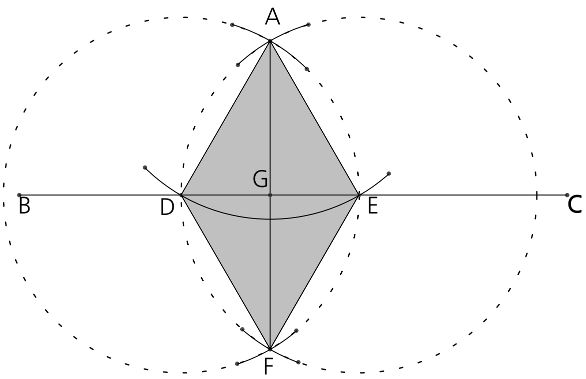
1 Introduction

We have already noted that the National Curriculum does not say how aspects of mathematics should be taught, nor, for example, do A/AS-level syllabuses. We are concerned that aspects of the geometry curriculum should not be taught in isolation. In this appendix we give examples of (a) how particular aspects of the geometry curriculum could be integrated within a particular theme, (b) where particular aspects of geometry could be linked with other areas of mathematics such as algebra and handling data and (c) where aspects of geometry could be linked with other subjects such as science, history and art.

2 Integration of aspects of geometry within a theme

A key to effective teaching of geometry is to combine experiential work with more formal argument in solving problems. As an example, consider straight edge and compass constructions, which are included in the Key Stage 3 programme of study. An approach sometimes encountered is to teach these constructions as a series of specific techniques, perhaps with some applications, such as finding the incentre or circumcentre of a triangle. A more fruitful approach could proceed as follows.

Certain questions are raised, such as '*What is the locus of a point which moves so that it is (a) an equal distance from two fixed points or (b) an equal distance from two fixed lines?*' These and similar questions are explored practically, for example, by getting pupils to stand in different places or using counters to represent points. In the course of this exploration, other questions will arise, for example, '*What is meant by the distance of a point from a line, and how is it found?*' (Dropping a perpendicular from a point to a line is another standard construction.)



Pupils become familiar with compasses as an instrument for constructing the locus of points which are a fixed distance from a fixed point. They recognise, for example when constructing a triangle given SSS, that the point of intersection of two circles is at specified distances from two fixed points.

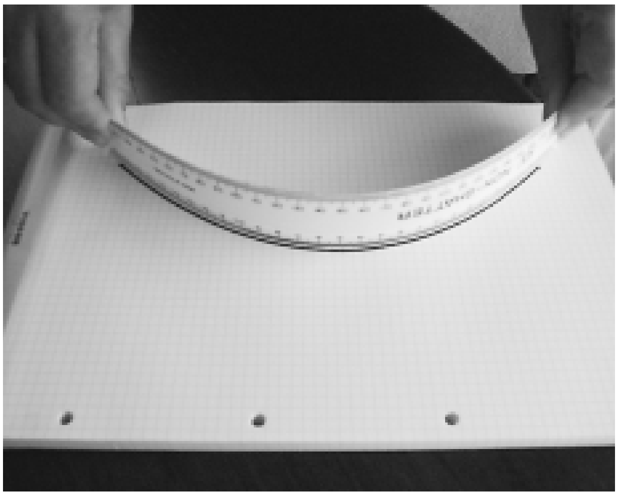
The practical exploration of loci described above motivates and provides a purpose for considering how to construct bisectors of lines and angles. The pupils' geometric awareness prompts consideration of how compasses can be used in construction.

The geometric aspects which are key to these and other basic constructions are those concerning the properties of the diagonals of a rhombus, namely that they bisect each other at right angles and also bisect the angles of the rhombus. If pupils have prior knowledge of these properties then, handled in an appropriate way, they can be used to approach the constructions as problem solving exercises, rather than a series of techniques to be learned.

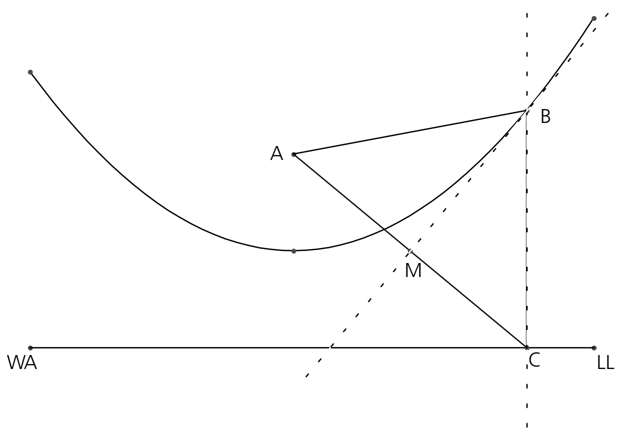
The benefits of this approach are that it develops problem solving skills, application of known results and reasons as to why particular techniques work. It also helps to integrate aspects of geometrical work (loci, properties of shapes, construction techniques) into a more powerful body of mathematics.

3 Integration of aspects of geometry with other areas of mathematics such as algebra and handling data

At Key Stage 4 pupils encounter quadratic functions in Ma2 *Number and algebra*. They also develop further ideas of construction, locus and coordinates in Ma3 *Shape, space and measures*. In Ma4 *Handling data* they collect data and represent it graphically. There are a number of familiar physical objects which appear to exhibit quadratic shape (i.e. that of a parabola) or its 3-D equivalent. These include bridges, bent rulers and satellite receiver dishes. Using modern technology images can be easily captured from the Internet, from photographs, or on digital cameras. Even without such technology curves can be traced and coordinates read off from a suitable grid.



The locus examples above can be extended to considering the path of an object which moves such that its distance from a fixed point is the same as that from a fixed line. For example in the classroom or playground a straight wall can be chosen as the fixed line. Then one pupil, called *A*, can stand about half way along the wall and say 2 metres away from it. Now the class can try to direct another pupil, called *B*, to move so that his/her distance from *A* is the same as his/her distance (measured perpendicularly) from the wall. In order to turn this into a construction suitable for use on paper, or with computer software, suppose *C* is any point on the wall. We need to construct the point *B* such that $AB = BC$. But we also know that *B* must lie on the perpendicular to the wall at *C*. Now if *ABC* is to be an isosceles triangle then we know that the perpendicular bisector of its base *AC* must pass through the vertex at *A*. Hence we just need to find the intersection of the perpendicular bisector of *AC* with the perpendicular to the wall at *C*. Now by taking several positions of *C* (or just by `dragging' *C* using dynamic geometry software) we can find the locus of *B*.



Using sophisticated language we can see that in the above construction the line (*directrix*) has been replaced by a line segment. This is the *domain* of the *independent variable C*. The distance of the point *A* (*focus*) from the wall is a *parameter* of the problem. The point *B* has been defined by constructions using the points *A*, *C* and the wall so that it is a *dependent variable*, and by letting *C* track through its domain we can find its *locus* with respect to *C*. So through geometry we can create images of functional relationships.

If we now take the wall as the *x*-axis, and its perpendicular through *A* (0, 2*a*) as the *y*-axis we can give the point *B* the coordinates (*x,y*) and use Pythagoras's theorem to find a relationship between them. The origin *O* (0,0) is the point on the wall nearest *A*. Let *D* be the point on *BC* so that *AD* is parallel to the wall. Then in the right angled triangled triangle *ADB* we have $AD = x$, $DB = y - 2a$ and $AB = BC = y$. Hence we have $y^2 = x^2 + (y - 2a)^2$, which simplifies to: $y = x^2/4a + a$, showing it is a quadratic function. We can also interpret the geometrical effects on the graph of $y = x^2$ of multiplying by a factor (1/4*a*) and adding the constant *a*.

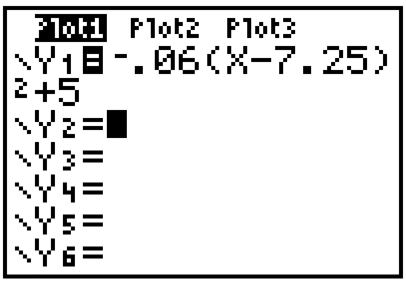
It also appears from the diagram that the perpendicular bisector of *AC* is a tangent to the parabola at *B*. Assuming this to be the case it is straightforward to derive the reflecting property of parabolas used in optical telescopes, and parabolic satellite dishes. Note: the lack of feasibility of a proof at this stage need not deter us from engaging with the activity. However, it is important not to gloss over such gaps in the logic but to emphasise them as unfinished business which will need to be resolved if the theory is to be watertight. At A-level the result can be proved using calculus to find the equation of the tangent to the parabola at *B*.

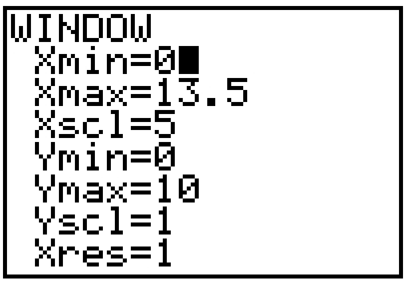
Another example uses an image as a source of data. Below there is a photograph of Sydney Harbour bridge. Tracing paper could be used to run over any of the curved sections from which coordinates could be read manually. Another way is to scan the photograph, or use a digital camera, to capture the image in picture editor software. As the cursor moves over the image a read-out of pixel coordinates is obtained automatically in the bottom right of the display.

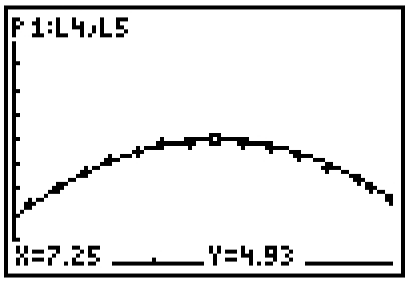
The table below shows the coordinates for a selection of points on the front lower curved arch.

x	58	152	252	348	440	533	627	725	817	918	1020	1122	1236	1285	1349
y	484	411	352	304	268	244	232	228	236	255	286	330	386	420	461

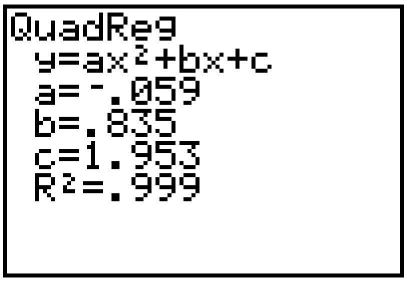
Of course the data are from an arbitrary origin and measured in pretty ghastly units. Data have been sampled for 15 of the 32 points where the vertical struts meet the lower front curved girder - which looks fairly like a parabola. The data can be transformed and displayed as a scattergram. A quadratic model can then be fitted by eye. For example, using a graphical calculator the data can be entered into lists L1 and L2. To transform the coordinates relative to an origin at the bottom left corner, which has pixel coordinates (0,721), 721-L2 can be stored in L3. Coordinates can be rescaled into units of, say, 100 pixels by storing L1/100 in L4 and L3/100 in L5. The scatterplot of L4 against L5 can then be drawn and quadratic functions superimposed by eye.

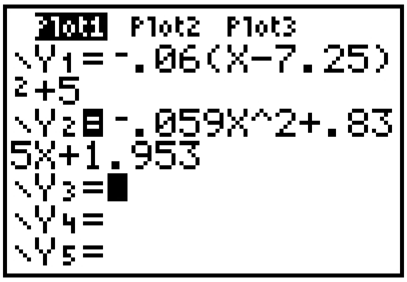


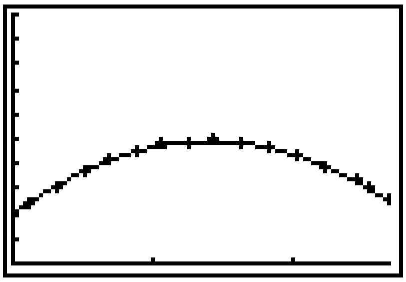




Many data handling packages, and graphical calculators, also provide the means of fitting models automatically. For example, quadratic regression gives a very good fit!







But it is also a nice exercise in algebra to equate the polynomial in the form: $ax^2 + bx + c$ with that in the form: $p(x - q)^2 + r$

This latter form is more convenient for modelling, and gives a live example of the general idea of transformations of functions f(x) in the form: $pf(qx + r) + s$. (Of course the transformation from $ax^2 + bx + c$ to $p(x - q)^2 + r$ is also the fundamental step in deriving the formula for the solution of a quadratic equation.)

Returning to the geometry, we note that here it is essential to get the aspect ratio right if the graph is in any way to match the photo! A nice exercise is to try to estimate the size of the new units in metres - zooming in on the original photo reveals a party of walkers nearing the flag at the top of the highest girder! Alternatively some research can be conducted, perhaps using the Internet, to find the span of the Sydney Harbour bridge. If we defined a measure called the 'bulge ratio', say, as b = (max(L5) - min(L5))/(max(L4) - min(L4)) - do you think that b is about the same (0.2) for all bridges of this type?

Try it on photographs of other bridges such as that across the Tyne in Newcastle. What about the cables on a suspension bridge - do they look like parabolas? Do they have a constant 'sag ratio'?

We can relate the image directly to geometric constructions, to test how well a parabola, or other curve, fits the bridge's shape. Here the image is pasted into a dynamic geometry package. Key points *A*, *B* and *C* are identified on the bridge. *A* and *C* are where the road meets the arch under question, and *B* is the highest point of the arch. The perpendicular through *B* to *AC* is constructed and *F* is taken as any point on it, (as in the next figure). *D* is the reflection of *F* in the line through *B* parallel to *AC*, and *GH* is part of the line parallel to *AC* through *D*. *P* is any point of *GH*. The perpendicular bisector of *PF* meets the perpendicular to *GH* at *P* in the point *Q*. The locus of *Q* as *P* varies is a parabola. *F* is the unique `control point' and as this is slid up and down so the shape of the parabola alters dynamically. There is only one position of *F* for which the parabola passes through *A* and *C*

An alternative geometric model for the curved arch could just be a circular arc through *A*, *B* and *C* - how would you find its centre? Can you detect any visible difference between the parabolic and circular models? Some mathematical analysis, as well as further images, can be found at http://www.brantacan.co.uk/ and many other images at: http://architecture.about.com/arts/architecture/

4 Integrating aspects of geometry with other subjects such as science, history and art

The 1996 OECD publication, *Changing the subject: innovations in science, mathematics and technology education*, contains accounts of a number of innovative projects in mathematics and science. An American project sought to integrate ideas in mathematics and science with the real world. Their basic tenet was that if they could not explain to students in the first lesson on a new subject why they were about to study it, then it had to be deleted from the curriculum. Sadly that project team could not find a suitable justification for teaching about conics, such as the parabola and ellipse, and so axed them from the course! In the examples above we have described one major reason for investigating parabolas and quadratics. Developments in optics, and the significance of the invention of the reflecting telescope, right up to the current interest in the Hubble telescope, are matters of considerable interest not only in science, but also in the history of ideas and now in the reality of mass worldwide communications. The contribution of people such as Galileo, Kepler and Newton to our understanding of cosmology and gravity are important aspects of a general education. That their work led to mathematical models such as the elliptic orbits of the planets around the Sun and the parabolic trajectory of a bullet from a gun are also important aspects of education.

The discovery of the laws of perspective in renaissance art and architecture is another source of productive links between mathematics, art and history. There are good materials on which to base such work, such as those produced for the Royal Institution's masterclasses by Sir Christopher Zeeman, (see Appendix 11), and the book by Dr JV Field referenced in Appendix 14.